transformation, because the original Eq. (1) can be used much more easily, provided that the correct values are used. By this consideration, it is unnecessary to indicate the stationary and moving frame as done by Adarkar. In conclusion, to my knowledge the problem in my previous note is mathematically well posed.

Reference

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Comment on "Validity of Integral Methods in MHD Boundary-Layer Analysis"

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HEYWOOD and Moffatt[1] have given a detailed and lucid discussion on the difficulties involved in applying the so-called integral methods to MHD problems. Here, we wish to supplement their work by emphasizing the fact that the difficulties are inherent in nature, and much more deeply rooted than usually suspected. We wish to exemplify this by the following observation.

Consider the MHD Rayleigh problem[2] governed by the following system:

$$\begin{cases} \partial u/\partial t = \nu(\partial^2 u/\partial y^2) + a^2(\partial b/\partial y) \\ \partial b/\partial t = \eta(\partial^2 b/\partial y^2) + (\partial u/\partial y) \\ t = 0: \quad b = u = 0 \\ y = 0: \quad u = 1, \quad b = 0 \quad \text{(perfectly insulating)} \\ y \to \infty: \quad u \to 0, \quad b \to 0 \end{cases}$$

It is not necessary to explain the meaning of the symbols used here, except to indicate that $a^2(\partial b/\partial y)$ and $\partial u/\partial y$ represent the interactions between the magnetic and the flow fields. Now if we integrate this system with respect to y from 0 to ∞, the resulting integral relations are

$$\frac{d}{dt}\int_0^\infty u\,dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{d}{dt}\int_0^\infty b\,dy = -\eta \left.\frac{\partial b}{\partial y}\right|_{y=0} - 1$$

The influence of the magnetic field on the flow field is lost completely, whereas that of the flow field on the magnetic field is over-simplified to a constant term 1. It is obvious that regardless of which trial curves one uses for u and b, the result will not be acceptable.†

It is interesting, however, to notice that the same problem, if the induced magnetic field is neglected because of small viscous diffusivity compared to the magnetic diffusivity, is governed by one single equation

$$\partial u/\partial t = \nu(\partial^2 u/\partial y^2) - cu$$

which yields the integral relation

$$\frac{d}{dt}\int_0^\infty u\,dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} - c\int_0^\infty u\,dy$$

In this equation, the interaction is decently represented. As a matter of fact, a trial solution of the form

$$u = \operatorname{erfc}\left[\frac{y}{2(\nu)^{1/2}\delta(t)}\right]$$

yields results that check very well with the exact solution for all y values at small times. For large time, the check is very good near the wall, as expected.

References

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† The two integral relations are correct, of course. It is only incorrect (not even as an approximation) here to try to obtain local behavior from them. In other words, it is true that the net effect of the interaction on the fluid as a whole is represented by the two constants 0 and 1; locally, however, this is simplifying the matter too much.

Comments on the Analysis of Free Vibration of Rotationally Symmetric Shells

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IN a recent paper,[1] Stodola's iterative method of finding eigenvalues has been proposed for the analysis of free vibration of shells of revolution. This method is being offered as an alternate to the multisegment free vibration method employed earlier.[2] The writer would like to make some comments on the merits of these two methods.

First of all, the multisegment free vibration method employed in Ref. 2 is not an "iterative" method in the same sense as Stodola's method. According to the multisegment method, the frequency equation of an arbitrary shell of revolution is obtained in the form of a determinant of a (4×4) matrix whose elements, for a given frequency, are determined by means of direct numerical integration of the differential equations. The roots of the frequency equation are found in the same way as the roots of any transcendental algebraic equation. In practice, the finding of the roots is a relatively simple matter, and it involves a systematic evaluation of the frequency equation at selected points within a given frequency interval. As soon as a sign change in the determinant is detected, the natural frequency is determined as accurately as desired by means of inverse interpolation. The point is that no "convergence" is involved in the sense of convergence of an assumed solution toward the actual solution, which is the basis of the usual "iterative" methods, such as Stodola's method.

As is well known,[3] Stodola's method (sometimes called the method of Stodola and Vianello) has been of great practical value in vibration and stability problems of beams. The method starts with a rough estimate of the deflection curve of the beam, from which a better estimate is obtained. It can be proved (Ref. 3, p. 201) that this iteration process is convergent for the lowest eigenvalue. Higher eigenvalues can be obtained by subtracting out from the assumed solution all preceding eigenfunctions.

In the opinion of the writer, the only reason for wanting to apply Stodola's method to shells of revolution is that instead of eight homogeneous solutions, which are needed to find one value of the frequency determinant by the multisegment method, only one particular solution per iteration need be calculated. However, whereas in the multisegment method

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the eight solutions give one exact value of the frequency determinant, the particular solution in Stodola's method is for only one iteration that must be repeated to achieve convergence. Therefore, the number of particular solutions needed in Stodola's method to obtain one natural frequency clearly depends on how closely the assumed solution matches the exact solution, and, contrary to the statement of Ref. 1, it seems to the writer quite impossible to say which one of the two methods will require a smaller number of solutions per one natural frequency.

Beyond the question of shorter execution time, the method presented in Ref. 1 has several serious disadvantages that should be considered before it is used. First of all, the method is capable of finding only the lowest mode plus a certain number of successive modes for each circumferential wave number. Exactly how many, is not clear. In Ref. 1 it is stated that, because of accumulation of errors in successive modes and limited computer storage, about ten modes may be obtained, although not more than three for each case are displayed in the paper. It would seem to the writer that the number of modes which can be obtained with the method of Ref. 1 again is related directly to the choice of the assumed solutions. This can be concluded from the detailed calculations presented in Ref. 3 on p. 205, where it is shown that if a poor initial guess of the displacements is made, then, because of accumulation of errors at each iteration, difficulties may arise already in the second mode.

Second, the method of Ref. 1 always requires the calculation of all the preceding modes, down to the fundamental mode, before a particular desired mode of free vibration can be obtained. This means that whenever it is required to examine a given frequency interval, it is necessary to calculate all the preceding modes, regardless of whether or not they are wanted.

The incapability to find the natural frequencies in a given frequency interval, without calculating the preceding modes and the limited number of modes that can be obtained, is perhaps the most serious disadvantage of the method of Ref. 1. For many applications, such as in the construction of a transient solution, more than ten modes are needed. For example, for a hemispherical shell subjected to a sudden pressure change, about eighteen modes are required in the transient solution.[4] For other applications, such as the determination of resonant frequencies of a shell of revolution within a frequency interval, the method would fail completely if the resonant frequencies are among those modes that are destroyed by accumulative errors.

In addition to these two severe limitations of the method of Ref. 1, there is another feature inherent in Stodola's method which could become a serious problem when the method is applied to free vibration of shells of revolution. According to Ref. 1, one has to provide intelligent guesses for four displacement functions over the complete meridian of the shell for every mode of free vibration. For a beam, where only one displacement function is needed, this may not be too difficult. For some simple shells, especially for which the exact solutions are known a priori, this may not be much of a problem either. It should be kept in mind, however, that the method is claimed to be "automatic" for an arbitrary shell of revolution. It seems to the writer that the procedure of guessing four different functions, which ought to resemble the actual mode, for each natural frequency can be rather difficult and make the method certainly somewhat less than "automatic."

By contrast, the multisegment method is capable of calculating any one or any number of natural frequencies, and all relevant variables of a mode within any frequency interval. The calculated values of the frequency equation can be made as accurate as desired. The calculation is straightforward and involves no guessing of any kind.

Finally, as a subscriber and reader of the AIAA Journal, the writer is somewhat concerned about the wasted space in Ref. 1 devoted to the fact that in one of his earlier papers in place of mode #3 he inadvertently inserted mode #4. Thus, the first three bending modes were included, but the first membrane mode, which actually is below the third bending mode, was omitted. The author of Ref. 1 apparently has tried to make the most of this bookkeeping error as if the sole contribution of this paper would be the discovery of mode 3 of a 60° spherical shell with a thickness-to-radius ratio of 0.05.

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Reply by Author to A. Kalnins

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IN reply to Kalnins' comments on the method of shell vibration analysis presented in Ref. 1, I should first of all like to clarify some points of semantics. He uses the term "multisegment method" to describe the method presented in Ref. 2. It should be pointed out that both methods, being based on the technique of forward integration, require shell segmentation in order to avoid round-off errors associated with the rapid growth of fundamental solutions. Therefore, in the following, the method in which the input to a given iteration step is an estimate of the displacement modes[1] is called the mode method, and the method in which the input to a given iteration step is an estimate of the frequency[2] is called the frequency method. I agree that the frequency method is not an iterative method in the same sense as the mode method. As Kalnins himself describes it, the frequency method is essentially a systematized process of trial and error, whereas the mode method is a true iteration method with proven convergence. However, for the purpose of discussion, here, as in Ref. 1, each determinant evaluation of the frequency method is referred to as an iteration step.

With the preceding in mind, I should like to correct Kalnins' misinterpretation of the following statement of Ref. 1. "Thus, other things being equal, one can expect a considerably shorter execution time per step in the mode method." When read in context, it is clear that the word "step" refers to a single iteration step and not the complete solution for a given vibration mode.

Kalnins' alleged disadvantages of the mode method may be summarized, for convenience, to be the following: 1) The number of iterations required to converge to a given mode depends on "intelligent" initial guesses for the four modal displacement functions. 2) The method is capable of finding only the first few lowest modes for each circumferential wave number. 3) No mode can be obtained without obtaining previously all the preceding lower modes with the same number of circumferential waves.

The refutation of these statements follows: 1) In the mode method no attempt need be made to speed up convergence by providing intelligent initial guesses for the displacement modes. In fact, for all modes the initial guess built into our program is simply $u = v = \chi = 0$ and $w =$ linear function of s. As a result, the rate of convergence de-

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